On Higher Toda Brackets

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0. In computing the 2-components of the homotopy groups of spheres, we need to observe the behaviour of Toda brakets in EHΔ-sequence. In Toda's book[2]we find, for example, the following formula

$$H\{E\alpha,\ E\beta,\ E\gamma\}_1=-\Delta^{-1}(\alpha\cdot\beta)\cdot E^2\gamma.$$

In this paper we generalize such a formula to higher Toda brackets. The main result is Theorem 7, which contains the following assersion as a special case: Under some conditions, $H\{E\alpha_0, E\alpha_1, ..., E\alpha_s\}_1$ and $(-1)^{(t-1)(s-1)}\{\Delta^{-1}\{\alpha_0,...,\alpha_{t-1}\}, E^t\alpha_t, ..., E^t\alpha_s\}_t$ have a common element. *I* believe that this is known to earlier workers in the field, although this does not seem to be in the literature.

1. In this paper we assume that all spaces are connected topological spaces with base points which have the same homotopy type as *CW*-complexes. Let FX be a free monoid generated by X-(base point *). We may identify FX with ΩEX, the loop space of the suspension space EX. FX has a sequence of subspaces

$$\cdots \supset F_{K+1}X \supset F_KX \quad \cdots \supset F_1X=X \supset F_0X=*,$$

where F_kX consists of all n-fold product of points of X. Note that $F_KX/F_{K-1}X \simeq \Lambda^KX$ (k-fold smash product).

The combinatorial extension $h_r:FX\longrightarrow F\Lambda^rX$ of a relative homeomorphism $h':(F_rX, F_{r-1}X) \to(\Lambda^rX, *)$ is defined by

$$h_r(x_1\cdots x_t)=\prod_\sigma h'(x_{\sigma(1)}\cdots x_{\sigma(r)}),$$

where σ runs through all strict monotonic function $\{1,..., k\}\longrightarrow\{1,...,t\}$ and the order of the product is lexicographic from the right (or left). The James-Hopf invariant $H_r:[EY,EX]\longrightarrow[EY,E\Lambda^rX]$ is defined by $\Omega_1^{-1}\cdot h_{r*}\cdot\Omega_1$, where $\Omega_1:[EY,EX]\longrightarrow[Y,FX]$ is an adjoint isomorphism.

We define an additve relation $\Delta_r:[E^2Y,\ E^rX]\longrightarrow[Y,F_{r-1}X]$ by the composite

$$[E^2Y, E\Lambda^rX]\xrightarrow{\Omega_1}[EY,F\Lambda^rX]\xrightarrow{h_{r*}}$$
$$[CY,Y;FX,F_{r-1}X]\xrightarrow{\partial}[Y,F_{r-1}X],$$

where ∂ is the boundary homomorphism.

Proposition 1. *Suppose given* $W\xleftarrow{\alpha}X, EX\xleftarrow{\beta}EY$ *and* $Y\xleftarrow{\gamma}Z$. *Then we have*

i) $H_r(E\alpha\circ\beta)=E\Lambda^r\alpha\circ H_r(\beta)$, *ii)* $H_r(\beta\circ E\gamma)=H_r(\beta)\circ E\gamma$.

proposition 2. *Suppose given* $W\xleftarrow{\alpha}X, E\Lambda^rX\xleftarrow{\beta}E^2Y$ *and*

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$Y \xleftarrow{r} Z$, *Then we have*

i) $F_{r-1}\alpha \circ \Delta_r(\beta) \subset \Delta_r(E\Lambda^r \alpha \circ \beta)$, *ii)* $\Delta_r(\beta) \circ r \subset \Delta_r(\beta \circ E^2 r)$.

2. Now we define the higher Toda braket. Our definition is slightly modified to that of G. Walker [3].

Suppose given a sequence of spaces and homotopy classes of maps.

$$X_0 \xleftarrow{\alpha_0} E^n X_1, \quad X_1 \xleftarrow{\alpha_1} X_2 \leftarrow \cdots \leftarrow X_s \xleftarrow{\alpha_s} X_{s+1}$$

where $s \geqslant 1$. The s-ary Toda braket $\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n$

is defined if there exists a defining system. A defining system for this bracket is a set of maps

$$X_i \xleftarrow{H^{ij}} E^{\varepsilon}(I_{i+1} \Lambda \cdots \Lambda I_{j-1} \Lambda X_j),$$

for each pair (i,j) such that $0 \leqslant i < j \leqslant s+1$, $(i,j) \neq (0, s+1)$, where $\varepsilon = \varepsilon(i)$ is equal to n if $i = 0$ and 0 if $i > 0$. Here the I_n are copies of the unit interval I. The maps H^{ij} are required to satisfy two conditions. (1) $H^{i,i+1}$ represents α_i for $0 \leqslant i \leqslant s$. (2) The following diagram is commutative for i,j,k such that $0 \leqslant i < j < k \leqslant s+1$.

$$\begin{array}{ccc} X_i & \xleftarrow{H^{ij}} & E^{\varepsilon}(I_{i+1} \Lambda \cdots \Lambda I_{j-1} \Lambda \cdots \Lambda I^{l}{}_{k-1} \Lambda X_k) \\ & \nwarrow^{H^{ij}} & \downarrow E^{\varepsilon}(1 \Lambda H^{jk}) \\ & & E^{\varepsilon}(I_{i+1} \Lambda \cdots \Lambda I_{j-1} \Lambda X_j) \end{array}$$

By the condition (2) there is a map

$$X_0 \xleftarrow{\tilde{H}^{0,s+1}} E^n(\partial(I_1 \Lambda \cdots \Lambda I_s) \Lambda X_{s+1}) = E^{n+s-1} X_{s+1},$$

which is defined by the following diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{\tilde{H}^{0,s+1}} & E^n(I_1 \Lambda \cdots \Lambda I^1_i \Lambda \cdots \Lambda I_s \Lambda X_{s+1}) \\ & \nwarrow^{H^{0,i}} & \downarrow E^n(1 \Lambda H^{i,s+1}) \\ & & E^n(I_1 \Lambda \cdots \Lambda I_{i-1} \Lambda X_i) \end{array}$$

The s-ary Toda bracket $\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n \subset [E^{n+s-1}X_{s+1}, X_0]$ consists of all homotopy classes of maps $(-1)^{n(s+1)}\tilde{H}^{0,s+1}$ obtain from defining systems. For convenience, if $s=1$ the s-ary Toda bracket is interpreted as the composite $\alpha_0 \circ E^n\alpha_1$. If n=o, we write simply $\{\alpha_0, \ldots, \alpha_s\}$

The following two propositions are essesntially due to G. Walker [3].

Proposition 3. *If* $\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n$ *is defined, the entire set* $\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n$ *can be obtained from defining systems which start with any fixed set of maps* $H^{i,i+1}$ *representing* $\alpha_i (1 \leqslant i \leqslant s)$.

Proposition 4. *If* $\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n$ *is defined, then the indeterminacy of this bracket is contained in the union of the* $(s-1)$*-ary Toda bracket*

$$\{(\alpha_0, \beta_0), E^n \begin{bmatrix} \alpha_1 \beta_1 \\ 0 \; E\alpha_2 \end{bmatrix}, \ldots, E^n \begin{bmatrix} \alpha_{s-2}\beta_{s-2} \\ 0 \quad E\alpha_{s-1} \end{bmatrix}, E^n \begin{bmatrix} \beta_{s-1} \\ E\alpha_s \end{bmatrix}\}_n,$$

taken over all homotopy classes $\beta_0 \in [E^{n+1}X_2, X_0]$ *and* $\beta_i \in [EX_{i+2}, X_i]$ $(0 < i \leqslant s-1)$ *for which the bracket is defined.*

3. Next we study the effect of E, H_r and Δ_r on the higher Toda bracket. It is easy to see that the following formula holds for $n \geqslant 0$.

$$(-1)^{s+1} E\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n \subset \{E\alpha_0, E^{n+1}\alpha_1, \ldots, E^{n+1}\alpha_s\}_{n+1}.$$

Proposition 5. *Let* $n \geqslant 1$. *Suppose given a sequence* $EX_0 \xleftarrow{\alpha_0} E^nX_1$, $X_1 \xleftarrow{\alpha_1} X_2 \leftarrow \ldots \leftarrow X_s \xleftarrow{\alpha_s} X_{s+1}$. *If* $\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n$ *is defined, then* $\{H_r(\alpha_0), E^n\alpha_1, \ldots, E^n\alpha_s\}_n$ *is defined and we have*

$H_r\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n \quad \{H_r(\alpha_0), E^n\alpha_1, \ldots, E^n\alpha_s\}_n$. *Proof is easy.*

Proposition 6. *Let* $n \geqslant 2$. *Suppose given* $E\Lambda^r X_0 \xleftarrow{\alpha_0} E^nX_1$, $X_1 \xleftarrow{\alpha_1} X_2 \leftarrow \ldots \leftarrow X_s \xleftarrow{\alpha_s} X_{s+1}$ *Assume that* $\{\Delta_r(\alpha_0), E^{n-2}\alpha_1, \ldots, E^{n-2}\alpha_s\}_{n-2}$ *is defined and any defining system* $\{K^{i,j}\}$ *for this bracket satisfies the condition*

$$i \circ K^{0,j} \simeq 0 \qquad \text{for } 1 \leqslant j \leqslant s,$$

where $i: F_{r-1}X_0 \longrightarrow FX_0$ *is the inclusion. Then* $\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n$ *is defined and* $\{\Delta_r(\alpha_0), E^{n-2}\alpha_1, \ldots, E^{n-2}\alpha_s\}_{n-2} \subset \Delta_r\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n$. Proof. By the condition, the map $i \circ K^{0,j}: E^{n-2}(I_1 \Lambda \ldots \Lambda I_{j-1} \Lambda X_j) \to FX_0$ is extended to

$$B: CE^{n-2}(I_1 \Lambda \ldots \Lambda I_{j-1} \Lambda X_j) \longrightarrow FX_0.$$

Then there is a map

$$H^{0,j}: E^n(I_1 \Lambda \ldots \Lambda I_{j-1} \Lambda X_j) \longrightarrow F_\Lambda X_0$$

such that

$$h_r \circ B = (\Omega_1 \circ H^{0,j}) \circ p,$$

where $p: CE^{n-2}(I_1 \Lambda \ldots \Lambda I_{j-1} \Lambda X_j) \longrightarrow E^{n-1}(I_1 \Lambda \ldots \Lambda I_{j-1} \Lambda X_j)$ is the projection. Define $H^{i,j} = K^{i,j} (0 < i < s+1)$ and $\{H^{i,j}\}$ is adefining system for $\{\alpha_0, E^n\alpha_1, \ldots, E^n\alpha_s\}_n$. Note that $\Delta_r(H^{0,j}) = K^{0,j}$. Now the proposition follows from Proposition 2.

Let $d: EF_{r-1}X \longrightarrow EX$ denote the adjoint map to the inclusion $i: F_{r-1}X \longrightarrow FX$.

Theorem 7. *Let* $s \geqslant t \geqslant 2$. *Suppose given* $\alpha_i: X_{i+1} \longrightarrow X_i (0 \leqslant i \leqslant s)$ *such that Toda brackets*

(a) $\{d_*(EF_{r-1}\alpha_0), EF_{r-1}\alpha_1, \ldots, EF_{r-1}\alpha_s\}_1$,

and

(b) $\{F_{r-1}\alpha_0, \ldots, F_{r-1}\alpha_{t-1}\}$

are defined. Assume that there is a defining system $\{H^{i,j}\}$ *for the bracket* (a) *and a defining system* $\{G^{i,j}\}$ *for the bracket* (b) *such that*

(*) $\Omega_1 \circ H^{0,j} = i \circ G^{0,j} \qquad 0 < j < t,$

$$H^{i,j}=G^{i,j} \qquad 0<i<j<t.$$

Then Toda bracket

(c) $\{\Delta^{-1}\{F_{r-1}\alpha_0,\ldots,F_{r-1}\alpha_{t-1}\}, E^tF_{r-1}\alpha_t,\ldots,E^tF_{r-1}\alpha_s\}_t$

is defined, and

$$H_r\{d_*(EF_{r-1}\alpha_0), EF_{r-1}\alpha_1,\ldots,EF_{r-1}\alpha_s\}_1$$

and

$$(-1)^{(t-1)(s-1)}\{\Delta^{-1}\{F_{r-1}\alpha_0,\ldots,F_{r-1}\alpha_{t-1}\}, E^tF_{r-1}\alpha_t,\ldots,E^tF_{r-1}\alpha_s\}$$

have a commomn element.

Proof. For the defining systems $\{H^{i,j}\}$, $\{G^{i,j}\}$, we construct a defining systems $\{K^{i,j}\}$, $0\leqq i<j\leqq s+1$, $i\neq 1,\ldots,t-1$, $(i,j)\neq(0,s+1)$, for the bracket (c).

For simplicity, put $Y_i=F_{r-1}X_i$. Let

$q:I_1\Lambda\cdots\Lambda\hat{I}_j\Lambda\cdots\Lambda I_{t-1}\Lambda Y_t\longrightarrow I_1\Lambda\cdots\Lambda I_{t-1}\Lambda Y_t$ be the inclusion and be the inclusion and

$p:I_1\Lambda\cdots\Lambda I_{t-1}\Lambda Y_t\longrightarrow E^{t-1}Y_t$

the projection. Since

$$h_r\circ(\Omega_1\circ H^{0,t})\circ q=h_r\circ(\Omega_1\circ H^{0,j})\circ(1\Lambda H^{j,t})$$
$$=h_r\circ i\circ G^{0,j}\circ(1\Lambda G^{j,t}) \quad =*,$$

there is a map

$K^{0,t}:E^tY_t\longrightarrow E\Lambda^rX_0$

such that $K^{0,t}$ represents

$\Delta_r^{-1}\{F_{r-1}\alpha_0,\ldots,F_{r-1}\alpha_{t-1}\}$.

For $k>t$, $K^{0,k}$ is similarly defined and for $t\leqq i<j\leqq s+1$ we define $K^{i,j}=H^{i,j}$. Then $\{K^{i,j}\}$ is a defining system for the bracket (c). Observe that

$$h_r\circ\Omega_1\circ H^{0,j}\circ(1\Lambda H^{j,s+1})=\Omega_1\circ K^{0,j}\circ E^{t-1}(1\Lambda G^{j,s+1})$$

for $j\geqslant t$ and

$$h_r\circ\Omega_1\circ H^{0,j}\circ(1\Lambda H^{j,s+1})=*$$

for $j<t$. This implies that

$$h_r\circ\Omega_1 H^{0,s+1}\simeq\Omega_1\circ\widetilde{K}^{0,s+1},$$

and the statement follows.

References

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